A more complete Exposition of those memorable series which are formed from the binomial coefficients*

Leonhard Euler

§1 I was led to the summation of these progressions especially by an appropriate notation[1] I introduced to represent the binomial coefficients of an arbitrary power succinctly. Of course, I exhibited the indefinite power of the binomial $(1+z)^n$ by means of the following series

$$(1+z)^n = 1 + \binom{n}{1}z + \binom{n}{2}zz + \binom{n}{3}z^3 + \binom{n}{4}z^4 + \text{etc.},$$

so that the coefficient of the power z^p is $\left(\frac{n}{p}\right)$, in which character expressed in the form of a fraction the upper number, n, denotes the exponent of the power itself, the lower number, p, on the other hand indicates, how often this coefficient was counted from the beginning. But it is known from the expansion of this product that it always is as follows

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n, \quad \binom{n}{2} = \frac{n}{1}\cdot\frac{n-1}{2}, \quad \binom{n}{3} = \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}$$

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[1]Euler refers to his paper “De mirabilibus proprietatibus unciarum, quae in evolutione binomii ad potestatem quamcunqua evecti occurrunt“. This is paper E575 in the Eneström-Index.

and in general

$$\left(\frac{n}{p}\right) = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-p+1}{p};$$

further, since the last term of the expanded binomial is z^n, it will be

$$\left(\frac{n}{n}\right) = 1;$$

and since the coefficients going backwards from the last term have the same structure as if one would start from the first term, it will be

$$\left(\frac{n}{n-1}\right) = \left(\frac{n}{1}\right), \quad \left(\frac{n}{n-2}\right) = \left(\frac{n}{2}\right) \quad \text{and in general} \quad \left(\frac{n}{n-p}\right) = \left(\frac{n}{p}\right).$$

Furthermore, hence it is also obvious that the value of this character $\left(\frac{n}{p}\right)$ will always go over into zero, as often as p was either a negative number or a positive number greater than n.

§2 Having explained these things I contemplated the series whose single terms are products of two binomial coefficients of an arbitrary power combined with each other in order, of which kind in general this progression is

$$s = \left(\frac{m}{0}\right)\left(\frac{n}{p}\right) + \left(\frac{m}{1}\right)\left(\frac{n}{p+1}\right) + \left(\frac{m}{2}\right)\left(\frac{n}{p+2}\right) + \left(\frac{m}{3}\right)\left(\frac{n}{p+3}\right) + \text{etc.},$$

until one finally gets to vanishing terms, as also the terms, which would precede the first, would vanish; and I showed that the sum of such a progression always is

$$\left(\frac{m+n}{m+p}\right) \quad \text{or even} \quad \left(\frac{m+n}{n-p}\right).$$

The proof of this theorem certainly seems to be of such a nature that it only holds for integer exponents m and n; but nevertheless I showed already that the same summation also holds for fractional exponents, if only the value of the character $\left(\frac{m+n}{m+p}\right)$ is defined correctly by known methods of interpolation.

§3 But this interpolation is most conveniently done by means of logarithmic integral formulas. For, it is known, if for the sake of brevity one puts $\log \frac{1}{x} = u$ and the following integrals are always taken from the lower limit $x = 0$ to the upper limit $x = 1$, that it will be as follows

$$\int u\partial x = 1, \quad \int uu\partial x = 1 \cdot 2, \quad \int u^3 \partial x = 1 \cdot 2 \cdot 3, \quad \int u^4 \partial x = 1 \cdot 2 \cdot 3 \cdot 4$$

and in general

$$\int u^p \partial x = 1 \cdot 2 \cdot 3 \cdot 4 \cdots p.$$

Furthermore, it will be

$$\int u^0 \partial x = 1.$$

But if the exponent p denotes an arbitrary negative integer number, the value of the integral $\int u^p \partial x$ will always be infinite. For, since in general it is

$$\int u^{p+1} \partial x = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (p+1) = (p+1) \int u^p \partial x,$$

it will vice versa be

$$\int u^p \partial x = \frac{1}{p+1} \int u^{p+1} \partial x.$$

Hence, if we take $p = -1$, this equation results

$$\int \frac{\partial x}{u} = \frac{1}{0} \int u^0 \partial x = \frac{1}{0} = \infty.$$

Furthermore, having taken $p = -2$ one will have

$$\int \frac{\partial x}{u^2} = -\frac{1}{1} \int \frac{\partial x}{u} = -\frac{1 \cdot 1}{1 \cdot 0} = \infty.$$

Hence it is plain that also all the following integrals become infinite. But whenever p denotes a fractional number, such an expansion cannot be true any longer, but we have to be content with the transcendental quantity which is expressed by the formula $\int u^p \partial x$. So it is known for a long time[2], if it was $p = -\frac{1}{2}$, that then it is

$$\int \frac{\partial x}{\sqrt{u}} = \sqrt{\pi},$$

[2] Euler refers to his paper "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt". This is paper E19 in the Eneström-Index.

while π denotes the circumference of the circle whose diameter is $= 1$. Therefore, hence applying the reduction mentioned before it will be

$$\int \partial x \sqrt{u} = \frac{1}{2}\sqrt{\pi}$$

and in like manner further it is

$$\int u^{\frac{3}{2}} \partial x = \frac{1 \cdot 3}{2 \cdot 2}\sqrt{\pi}$$

and further

$$\int u^{\frac{5}{2}} \partial x = \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2}\sqrt{\pi}$$

and

$$\int u^{\frac{7}{2}} \partial x = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 2}\sqrt{\pi}$$

etc.

But whenever p is a fraction of such a kind, whose denominator is greater than 2, then the values of integral formulas of this kind are reduced to higher transcendental quadratures.

§4 Having explained all these things in advance the summation of the progression mentioned above can be exhibited by means of integral formulas of this kind; for, is it easily understood that it will be

$$s = \left(\frac{m+n}{m+p}\right) = \frac{\int u^{m+n} \partial x}{\int u^{m+p} \partial x \cdot \int u^{n-p} \partial x}.$$

For, if m, n and p were positive integer numbers, it will also be

$$\int u^{m+n} \partial x = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (m+n).$$

In like manner it will be

$$\int u^{m+p} \partial x = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (m+p),$$
$$\int u^{n-p} \partial x = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-p);$$

hence it follows that it will be

$$\frac{\int u^{m+n}\partial x}{\int u^{m+p}\partial x} = (m+p+1)(m+p+2)\cdots(m+n),$$

where the number of factors is $= n-p$, which written in reverse order are of course

$$(m+n)(m+n-1)(m+n-2)\cdots(m+p+1).$$

But if this product is additionally divided by

$$\int u^{n-p}\partial x = 1\cdot 2\cdot 3\cdots(n-p),$$

where the number of factors likewise is $n-p$, one will find

$$\frac{\int u^{m+n}\partial x}{\int u^{m+p}\partial x\cdots\int u^{n-p}\partial x} = \frac{m+n}{1}\cdot\frac{m+n-1}{2}\cdot\frac{m+n-2}{3}\cdot\frac{m+n-3}{4}\cdots\frac{m+p+1}{n-p},$$

and this form obviously is the value of this character $\left(\frac{m+n}{n-p}\right)$, which likewise denotes the sum s in question. Although this proof seems to be restricted to integer numbers, nevertheless by the principle of continuity this expression exhibited by means of integral formulas must be true, whatever fractional numbers are taken for the letters m, n and p.

§5 These things almost reduce to those I published not that long ago on the summation of progressions of this kind[3]. But now it is propounded to me to find the same sums using very different method, of which I have given already several specimens[4]; this way not only the summations given here will be confirmed and illustrated, but also for the cases of fractional exponents the curves will be found, on whose quadratures the summations depend, whereas before these sums were expressed by means of quadratures of transcendental curves, such that this method will have the greatest use in the field of Analysis; but this method is based on the well known reduction of integral formulas, which I will use to our advantage in the following Lemmas.

[3]Euler refers to his paper "De vero valore formulae integralis $\int dx\,(\log x)^n$ a termino $x=0$ usque ad terminum $x=1$ extensae". This is the paper E662 in the Eneström-Index.

[4]Euler refers to his papers "Observationes circa integralia formularum $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ posito post integrationem $x = 1$" and "Comparatio valorum formulae integralis $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ a termino $x = 0$ usque ad $x = 1$ extensae". These are the papers E321 and E640 in the Eneström-Index.

Lemma 1

§6 *If one puts*

$$V = x^a(1-x^b)^{\frac{c}{b}},$$

it will be

$$\log V = a\log x + \frac{c}{b}\log(1-x^b)$$

and by differentiating

$$\frac{\partial V}{V} = \frac{a\partial x}{x} - \frac{cx^{b-1}\partial x}{1-x^b};$$

hence by multiplying by V and integrating again we will get to this reduction

$$V = x^a(1-x^b)^{\frac{c}{b}} = a\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} - c\int x^{a+b-1}\partial x(1-x^b)^{\frac{c-b}{b}}$$

and hence we deduce the two following reductions.

$$\text{I.}\quad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{1}{a}x^a(1-x^b)^{\frac{c}{b}} + \frac{c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1},$$

$$\text{II.}\quad \int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = -\frac{1}{c}x^a(1-x^b)^{\frac{c}{b}} + \frac{a}{c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}}.$$

Corollary

§7 Hence if these integrals are to be extended from lower limit $x=0$ to the upper limit $x=1$ and all exponents a, b etc. were positive, then in each of the two reductions the algebraic term is thrown out of the calculation completely, since it vanishes so for $x=0$ as for $x=1$, and these two reductions will then look as follows

$$\text{I.}\quad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

and

$$\text{II.}\quad \int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a}{c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}}.$$

But if the exponents a, b and c were not positive, the algebraic or absolute term cannot be omitted in these reductions, since it becomes infinite either in the case $x = 0$ or in the case $x = 1$. But here the exponent b can always be considered to be positive.

LEMMA 2

§8 *Having, as before, put*

$$V = x^a(1-x^b)^{\frac{c}{b}},$$

if both fractions we found for $\frac{\partial V}{V}$ are reduced to the ones with the common denominator, we will have

$$\frac{\partial V}{V} = \frac{a\partial x - (a+c)x^b\partial x}{x(1-x^b)}.$$

But if we now again multiply by V and integrate, we will get to this equation

$$V = x^a(1-x^b)^{\frac{c}{b}} = a\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} - (a+c)\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1},$$

whence these two equations follow

$$\text{I.}\quad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{1}{a}x^a(1-x^b)^{\frac{c}{b}} + \frac{a+c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

and

$$\text{II.}\quad \int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{-1}{a+c}x^a(1-x^b)^{\frac{c}{b}} + \frac{a}{a+c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1}.$$

COROLLARY

§9 If these integrals, as we will always assume in the following, must be extended from lower limit $x = 0$ to the upper limit $x = 1$ and the exponents a

etc. were positive, it will be possible to omit the absolute terms, so that then the following reductions will hold

$$\text{I.} \qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a+c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

and

$$\text{II.} \qquad \int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a}{a+c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1}.$$

LEMMA 3

§10 *Having put again*

$$V = x^a(1-x^b)^{\frac{c}{b}},$$

since we found above

$$\frac{\partial V}{V} = \frac{a\partial x-(a+c)x^b\partial x}{x(1-x^b)},$$

if we here write $(a+c)\partial x - c\partial x$ instead of the first term $a\partial x$, it will be

$$\frac{\partial V}{V} = \frac{\partial x(a+c)}{x} - \frac{c\partial x}{x(1-x^b)},$$

which equation multiplied by V and integrated yields

$$V = x^a(1-x^b)^{\frac{c}{b}} = (a+c)\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} - c\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1},$$

whence the two following reductions are obtained

$$\text{I.} \qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{1}{a+c}x^a(1-x^b)^{\frac{c}{b}} + \frac{a}{a+c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

and

$$\text{II.} \qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = -\frac{1}{c}x^a(1-x^b)^{\frac{c}{b}} + \frac{a+c}{c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}}.$$

COROLLARY

§11 Therefore, if the exponents a etc. were positive and the integrals must be extended from $x = 0$ and $x = 1$, having omitted the absolute term these reductions will result

$$\text{I.} \qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{c}{a+c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

and

$$\text{II.} \qquad \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a+c}{c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}}.$$

PROBLEM 1

§12 *If, having extended the integration from $x = 0$ to $x = 1$, the exponents a and c were positive and the value of this integral formula was known*

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \Delta,$$

in terms of the same to express all integral formulas contained in this general form

$$\int x^{a+ib-1}\partial x(1-x^b)^{\frac{c}{b}-1}.$$

SOLUTION

Here the reduction given in the second corollary of the second lemma is to be used, which is

$$\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a}{a+c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1},$$

where we want to augment the exponent a continuously by the number b; and since by assumption it is

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \Delta,$$

the following reductions will be found

$$\begin{aligned}
\text{I. } \int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} &= \frac{a}{a+c}\cdot\Delta,\\
\text{II. } \int x^{a+2b-1}\partial x(1-x^b)^{\frac{c}{b}-1} &= \frac{a}{a+c}\cdot\frac{a+b}{a+b+c}\cdot\Delta,\\
\text{III. } \int x^{a+3b-1}\partial x(1-x^b)^{\frac{c}{b}-1} &= \frac{a}{a+c}\cdot\frac{a+b}{a+b+c}\cdot\frac{a+2b}{a+2b+c}\cdot\Delta,\\
\text{IV. } \int x^{a+4b-1}\partial x(1-x^b)^{\frac{c}{b}-1} &= \frac{a}{a+c}\cdot\frac{a+b}{a+b+c}\cdot\frac{a+2b}{a+2b+c}\cdot\frac{a+3b}{a+3b+c}\cdot\Delta
\end{aligned}$$

etc.,

the structure of which progression immediately meets the eye.

COROLLARY 1

§13 If it was $a+c=b$ and hence $c=b-a$, it will be

$$\Delta=\int\frac{x^{a-1}\partial x}{(1-x^b)^{\frac{a}{b}}},$$

where it is to be noted that not only the exponent a must be positive, but also smaller than b, since also c must be positive. But this formula can conveniently be reduced to the quadrature of the circle; in order to show this just put

$$\frac{x}{\sqrt[b]{1-x^b}}=y,$$

such that for $x=0$ it also is $y=0$; but for $x=1$ it will be $y=\infty$; but then it will be

$$\Delta=\int\frac{y^a\partial x}{x}$$

and having raised y to the power of b it will be

$$y^b=\frac{x^b}{1-x^b},$$

whence it is found

$$x^b=\frac{y^b}{1+y^b},$$

and hence by taking logarithms it will be

$$b \log x = b \log y - \log(1 + y^b),$$

whence by differentiating it is concluded

$$\frac{\partial x}{x} = \frac{\partial y}{y} - \frac{y^{b-1}\partial y}{1 + y^b} = \frac{\partial y}{y(1 + y^b)};$$

having substituted this value it will be

$$\Delta = \int \frac{y^{b-1}\partial y}{1 + y^b};$$

since this integral must be extended from $y = 0$ to $y = \infty$, its value, as I showed on another occasion[5] is

$$= \frac{\pi}{b \sin \frac{a\pi}{b}}.$$

COROLLARY 2

§14 Therefore, if in general we set $c = b - a$, so that it is

$$\Delta = \frac{\pi}{b \sin \frac{a\pi}{b}},$$

the single reductions found in the problem will be as follows:

$$\text{I.} \quad \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1 - x^b)^a}} = \frac{a}{b} \cdot \Delta,$$

$$\text{II.} \quad \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1 - x^b)^a}} = \frac{a}{b} \cdot \frac{a+b}{2b} \cdot \Delta,$$

$$\text{III.} \quad \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1 - x^b)^a}} = \frac{a}{b} \cdot \frac{a+b}{3b} \cdot \frac{a+2b}{3b} \cdot \Delta,$$

$$\text{IV.} \quad \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1 - x^b)^a}} = \frac{a}{b} \cdot \frac{a+b}{4b} \cdot \frac{a+2b}{3b} \cdot \frac{a+3b}{4b} \cdot \Delta$$

etc.,

where it is evident that the coefficients of Δ completely identical to the binomial coefficients of $(1 - x^b)^{-\frac{a}{b}}$, which by means of an expansion yields

$$1 + \frac{a}{b}x^b + \frac{a}{b} \cdot \frac{a+b}{2b}x^{2b} + \frac{a}{b} \cdot \frac{a+b}{2b} \cdot \frac{a+2b}{3b}x^{3b} + \text{etc.}$$

[5] Euler refers to his paper "De expressione integralium per factores". This is paper E254 in the Eneström-Index.

Problem 2

§15 *If for the sake of brevity we put*

$$(1-x^b)^{-\frac{a}{b}} = 1 + Ax^b + Bx^{2b} + Cx^{3b} + \text{etc.},$$

so that it is

$$A = \frac{a}{b}, \quad B = \frac{a}{b}\cdot\frac{a+b}{2b}, \quad C = \frac{a}{b}\cdot\frac{a+b}{2b}\cdot\frac{a+2b}{3b} \quad \text{etc.},$$

to investigate the sum of this series

$$S = 1 + A^2 + B^2 + C^2 + D^2 + etc.$$

Solution

Therefore, since it is

$$(1-x^b)^{-\frac{a}{b}} = 1 + Ax^b + Bx^{2b} + Cx^{3b} + \text{etc.},$$

let us multiply both sides by

$$\frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

and by integrating we will have

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{2a}}} = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} + A\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} + \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$
$$+C\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} + \text{etc.}$$

But we taught how to express these integral formulas by means of the quantity Δ; if these values are substituted, we will get to the following series:

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{2a}}} = \Delta + A\cdot\frac{a}{b}\cdot\Delta + B\cdot\frac{a}{b}\cdot\frac{a+b}{2b}\cdot\Delta + C\cdot\frac{a}{b}\cdot\frac{a+b}{2b}\cdot\frac{a+2b}{3b}\cdot\Delta + \text{etc.},$$

which series is obviously reduced to this one

$$\Delta(1 + A^2 + B^2 + C^2 + D^2 + \text{etc.}),$$

whence the sum in question of our series will be

$$S = \frac{1}{\Delta}\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{2a}}},$$

while it is

$$\Delta = \frac{\pi}{b \sin\frac{a\pi}{b}}.$$

COROLLARY 1

§16 Let us first consider the case, in which it is $b = 2$, and since one has to take $a < b$, let $a = 1$, whence it is $\Delta = \frac{\pi}{2}$; but then we will have for the series itself

$$A = \frac{1}{2}, \quad B = \frac{1}{2}\cdot\frac{3}{4}, \quad c = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}, \quad D = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8} \qquad \text{etc.},$$

and the sum of the series

$$1 + A^2 + B^2 + C^2 + D^2 + \text{etc.}$$

will be

$$S = \frac{2}{\pi}\int \frac{\partial x}{1-xx}.$$

But on the other hand it is

$$\int \frac{\partial x}{1-xx} = \frac{1}{2}\log\frac{1+x}{1-x},$$

which expression having put $x = 1$ becomes infinite. But the sum of this series

$$1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 + \left(\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\right)^2 + \text{etc.}$$

is indeed infinitely large, as I showed on another occasion[6].

[6]Euler refers to E575 again.

COROLLARY 2

§17 Let us also consider the case $b = 3$ and let us take $a = 1$ that the exponent is $\frac{2a}{b} = \frac{2}{3}$ and hence still smaller than 1. Therefore, in this case for the series itself we will have

$$A = \frac{1}{3}, \quad B = \frac{1}{3} \cdot \frac{4}{6}, \quad C = \frac{1}{3} \cdot \frac{4}{6} \cdot \frac{7}{9}, \quad D = \frac{1}{3} \cdot \frac{4}{6} \cdot \frac{7}{9} \cdot \frac{10}{12},$$
$$E = \frac{1}{3} \cdot \frac{4}{6} \cdot \frac{7}{9} \cdot \frac{10}{12} \cdot \frac{13}{15} \quad \text{etc.},$$

and because of

$$\Delta = \frac{2\pi}{3\sqrt{3}}$$

the sum of the series

$$1 + A^2 + B^2 + C^2 + \text{etc.}$$

will be

$$S = \frac{3\sqrt{3}}{\pi} \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}},$$

which therefore can now be expressed by means of the quadrature of an algebraic curve, whose abscissa x corresponds to the ordinate

$$y = \frac{1}{\sqrt[3]{(1-x^3)^2}},$$

for which case the method given first yields a quadrature of a transcendental curve.

SCHOLIUM

§18 This expression for the sum of the series

$$1 + A^2 + B^2 + C^2 + \text{etc.}$$

cannot hold, if the exponent a was not positive, in which case therefore the power of the binomial $1 - x^b$ is negative and hence the series $1 + A^2 + B^2 + C^2 + \text{etc.}$ becomes an infinite series. Therefore, hence for the coefficients of the binomial raised to a positive power nothing can be concluded, although nevertheless this case is immediately clear using the first method. Further, since the sum of this series was found to be

$$S = \frac{1}{\Delta}\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{2a}}},$$

while

$$\Delta = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

it is evident, if it would be $a = b$, in which case it would be $A = 1$, $B = 1$, $C = 1$ etc., that then the sum of the series of squares will obviously be infinite, which would happen even more, if it would be $a > b$. Yes, even, if it was $2a = b$ or $a = \frac{1}{2}b$, in the first corollary we saw that also this sum is infinite. Therefore, the sum found here is restricted to this rigid boundaries 1°. $a > 0$ and 2°. $a < \frac{1}{2}b$. But how hence the sum can even be defined, whenever a is a negative number, we will see in the following.

PROBLEM 3

§19 *If, as before, it still is*

$$(1-x^b)^{-\frac{a}{b}} = 1 + Ax^b + Bx^{2b} + Cx^{3b} + \text{etc.}$$

and additionally one puts

$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \text{etc.},$$

so that it is

$$\mathfrak{A} = \frac{\alpha}{b}, \quad \mathfrak{B} = \frac{\alpha}{b}\cdot\frac{\alpha+b}{2b}, \quad \mathfrak{C} = \frac{\alpha}{b}\cdot\frac{\alpha+b}{2b}\cdot\frac{\alpha+2b}{3b} \quad \text{etc.},$$

to find the sum of the series composed of these two series

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \mathfrak{D}D + \text{etc.}$$

SOLUTION

Having put, as in the preceding example,

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \Delta,$$

so that it is

$$\Delta = \frac{\pi}{b \sin \frac{a\pi}{b}},$$

if it was $a > 0$, of course, the reductions applied here will give

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = A\Delta,$$
$$\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = B\Delta,$$
$$\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = C\Delta,$$
$$\int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = D\Delta$$

etc.

Therefore, because it is

$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \text{etc.},$$

if we multiply both sides by

$$\frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

and integrate from the lower limit $x = 0$ to the upper limit $x = 1$, we will get to the following series:

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = \Delta + \mathfrak{A}A\Delta + \mathfrak{B}B\Delta + \mathfrak{C}C\Delta + \text{etc.},$$

which is the series in question itself multiplied by Δ, and hence its sum $= \Delta S$. Therefore, hence we vice versa conclude that it will be

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \text{etc.} = \frac{1}{\Delta}\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}}.$$

but this summation can likewise only be true, if $a > 0$. But on the other hand the exponent α is not restricted here; hence it will possible to assume so negative as positive numbers for it. Here only this is to be observed: If it was not $a + \alpha < b$, the sum of the propounded series is always infinitely large.

COROLLARY 1

§20 Since a must always be contained within the limits 0 and b, let us take $b = 2$ and one has to take $a = 1$, whence it is

$$A = \frac{1}{2}, \quad B = \frac{1 \cdot 3}{2 \cdot 4}, \quad C = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}, \quad D = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \quad \text{etc.};$$

furthermore, we will have

$$\Delta = \int \frac{\partial x}{\sqrt{1 - xx}} = \frac{\pi}{2}.$$

Therefore, hence, whatever value is attributed to α, the sum of the series in question

$$A = 1 + \frac{1}{2}\mathfrak{A} + \frac{1 \cdot 3}{2 \cdot 4}\mathfrak{B} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\mathfrak{C} + \text{etc.}$$

will be

$$S = \frac{2}{\pi} \int \frac{\partial x}{\sqrt{(1 - xx)^{1+\alpha}}}.$$

Hence it is plain, as long as it was $1 + \alpha < 2$ and hence $\alpha < 1$, that the sum will always be finite.

COROLLARY 2

§21 Therefore, while it still is $a = 1$ and $b = 2$, since it must be $\alpha < 1$, let us expand some cases

I. Let $\alpha = 0$.

Hence it will be

$$\mathfrak{A} = 0, \quad \mathfrak{B} = 0, \quad \mathfrak{C} = 0 \quad \text{etc.}$$

and so the series to be summed will be $S = 1$, but our formula yields

$$S = \frac{2}{\pi} \int \frac{\partial x}{\sqrt{1 - xx}}.$$

But on the other hand our formula yields

$$\int \frac{\partial x}{\sqrt{1 - xx}} = \frac{\pi}{2},$$

whence it is $S = 1$, what agrees extraordinarily with the value found before.

II. Let $\quad \alpha = -1.$

In this case it will be

$$\mathfrak{A} = -\frac{1}{2}, \quad \mathfrak{B} = -\frac{1 \cdot 1}{2 \cdot 4}, \quad \mathfrak{C} = -\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \quad \text{etc.},$$

whence the series to be summed is

$$S = 1 - \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1 \cdot 1}{2 \cdot 4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} - \text{etc.}$$

On the other hand our integral formula yields

$$S = \frac{2}{\pi} \int \frac{\partial x}{\sqrt{(1-xx)^0}} = \frac{2}{\pi},$$

which sum agrees extraordinarily with the one we found from the logarithmic integral formulas.

III. Let $\quad \alpha = -2.$

Here it will be

$$\mathfrak{A} = -1, \quad \mathfrak{B} = 0, \quad \mathfrak{C} = 0 \quad \text{etc.}$$

The series to be summed will hence be

$$S = 1 - \frac{1}{2} - \frac{1}{2},$$

our integral formula yields

$$S = \frac{2}{\pi} \int \partial x \sqrt{1-xx}.$$

But from the corollary of the third lemma we have this reduction

$$\int x^{a-1} \partial x (1-x^b)^{\frac{c}{b}} = \frac{c}{a+c} \int x^{a-1} \partial x (1-x^b)^{\frac{c}{b}-1},$$

which applied to our case by putting $a = 1$, $b = 2$, $c = 1$ gives

$$\int \partial x \sqrt{1-xx} = \frac{1}{2} \int \frac{\partial x}{\sqrt{1-xx}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4};$$

therefore, hence it is $S = \frac{1}{2}$.

IV. Let $\quad \alpha = -3.$

In this case it will therefore be

$$\mathfrak{A} = -\frac{3}{2}, \quad \mathfrak{B} = +\frac{3 \cdot 1}{2 \cdot 4}, \quad \mathfrak{C} = \frac{3 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 6}, \quad \mathfrak{D} = \frac{3 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 8}, \quad \mathfrak{E} = \frac{3 \cdot 1 \cdot 1 \cdot 3 \cdot \ 5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \quad \text{etc.},$$

whence the series to be summed will be

$$S = 1 - \frac{1}{2} \cdot \frac{3}{2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{3 \cdot 1}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{3 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{3 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 8} + \text{etc.};$$

but our integral formula on the other hand yields

$$S = \frac{2}{\pi} \int \partial x (1 - xx).$$

But for the set limits of the integration it is

$$\int \partial x (1 - xx) = \frac{2}{3},$$

whence the sum in question will be

$$S = \frac{4}{3\pi}.$$

V. Let $\quad \alpha = -4.$

Therefore, in this case it will be

$$\mathfrak{A} = -2, \quad \mathfrak{B} = 1, \quad \mathfrak{C} = 0, \quad \mathfrak{D} = 0 \quad \text{etc.},$$

whence the series to be summed will be

$$S = 1 - 1 + \frac{1 \cdot 3}{2 \cdot 4} = \frac{1 \cdot 3}{2 \cdot 4} = \frac{3}{8};$$

but the integral formula on the other hand yields

$$S = \frac{2}{\pi} \int \partial x \sqrt{(1 - xx)^3} = \frac{2}{\pi} \int \partial x (1 - xx)^{\frac{3}{2}}.$$

But now by the reduction applied in case III., having taken $a = 1$, $b = 2$ and $c = 3$, we will have

$$\int \partial x (1 - xx)^{\frac{3}{2}} = \frac{3}{4} \int \partial x \sqrt{1 - xx}.$$

But we saw that it is

$$\int \partial x \sqrt{1 - xx} = \frac{\pi}{4},$$

whence it will be

$$\int \partial x (1 - xx)^{\frac{3}{2}} = \frac{3\pi}{16},$$

whence the sum in question is calculated to be $S = \frac{3}{8}$; this also agrees to the value found before.

Problem 4

§22 *While so the Latin letters A, B, C, D etc. as the Germanic letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. retain the same values we assigned to them in the preceding problem, to investigate the sum of the following series composed of them*

$$\begin{aligned} S' &= A + \mathfrak{A}B + \mathfrak{B}C + \mathfrak{C}D + \mathfrak{D}E + \text{etc.}, \\ S'' &= B + \mathfrak{A}C + \mathfrak{B}D + \mathfrak{C}E + \mathfrak{D}F + \text{etc.}, \\ S''' &= C + \mathfrak{A}D + \mathfrak{B}E + \mathfrak{C}F + \mathfrak{D}F + \text{etc.} \end{aligned}$$

etc.

Solution

Having again put

$$\Delta = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1 - x^b)^a}},$$

we saw in the preceding problem that it is

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = A\Delta,$$
$$\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = B\,\Delta,$$
$$\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = C\,\Delta$$
etc.

Since now we put

$$(1-x^b)^{\frac{-\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + +\mathfrak{C}x^{3b} + \text{etc.},$$

let us immediately multiply both sides by

$$\frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

and by integrating we will obtain the following form.

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = A\Delta + \mathfrak{A}B\Delta + \mathfrak{B}C\Delta + \mathfrak{C}D\Delta + \mathfrak{D}E\Delta + \text{etc.},$$

which series obviously is $= \Delta S'$. Therefore, we hence conclude that it will be

$$S' = \frac{1}{\Delta}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}},$$

where it as up to now is

$$\Delta = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{\pi}{b\sin\frac{a\pi}{b}}.$$

To find the second series multiply that form

$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \text{etc.},$$

by the formulas

$$\frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}},$$

and having integrated term by term we will be led to the following form

$$\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = B\Delta + \mathfrak{A}C\Delta + \mathfrak{B}E\Delta + \mathfrak{C}E\Delta + \mathfrak{D}F\Delta + \text{etc.},$$

which is the second propounded series multiplied by Δ and hence $\Delta S''$, whence we conclude that it will be

$$S'' = \frac{1}{\Delta}\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}}.$$

From these things it is already understood, how the sums of all propounded series can be assigned; for, it is as follows

$$\begin{aligned} S''' &= \frac{1}{\Delta}\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} \\ S^{\mathrm{IV}} &= \frac{1}{\Delta}\int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} \\ S^{\mathrm{V}} &= \frac{1}{\Delta}\int \frac{x^{a+5b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} \\ &\text{etc.} \end{aligned}$$

and hence further one concludes that in general it will be

$$S^{(n)} = \frac{1}{\Delta}\int \frac{x^{a+nb-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}}$$

But here the condition prescribed above still holds that the value of the exponent a must be contained within the limits 0 and b. Further, it is to be noted on the exponent α that these sums can only be finite, if it is $\alpha + a < b$. Therefore, let us now see, how these summations must be accommodated to other values of the exponent a.

PROBLEM 5

§23 *If the exponent a was a negative number, nevertheless smaller than b, so that it is $a + b > 0$, to find the sum of the series*

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \mathfrak{D}D + \text{etc.},$$

where the capital letters have the same values as before, namely

$$(1-x^b)^{-\frac{a}{b}} = 1 + Ax^b + Bx^{2b} + Cx^{3b} + Dx^{4b} + \text{etc.},$$
$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \mathfrak{D}x^{4b} + \text{etc.}$$

Solution

Since the exponent $a+b$ is positive, let us start the reductions exhibited above in § 14 from the second one and put

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \Delta',$$

whence the reductions exhibited above will be reduced to the following ones

$$\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{a+b}{2b}\Delta' = \frac{b}{a}B\Delta'$$
$$\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{a+b}{2b}\cdot\frac{a+2b}{3b}\Delta' = \frac{b}{a}C\Delta'$$
$$\int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{a+b}{2b}\cdot\frac{a+2b}{3b}\cdot\frac{a+3b}{4b}\Delta' = \frac{b}{a}D\Delta'$$
$$\int \frac{x^{a+5b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{a+b}{2b}\cdot\frac{a+2b}{3b}\cdot\frac{a+3b}{4b}\cdot\frac{a+4b}{5b}\Delta' = \frac{b}{a}E\Delta'$$

etc.

Having noted these things in advance let us consider the equation

$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \mathfrak{D}x^{4b} + \text{etc.},$$

which we want to multiply by

$$\frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

and want to integrate, and we will obtain the following equation:

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} - \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

$$= \mathfrak{A}\Delta' + \frac{b}{a}\mathfrak{B}B\Delta' + \frac{b}{a}\mathfrak{C}C\Delta' + \frac{b}{a}\mathfrak{D}D\Delta' + \text{etc.},$$

where we want to write $\frac{b}{a}\mathfrak{A}A\Delta'$ instead of the first term $\mathfrak{A}A$ that the series is reduced to this form

$$\frac{b}{a}\Delta'(\mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \mathfrak{D}D + \text{etc.}) = \frac{b}{a}\Delta'(S-1).$$

But since the exponent a is supposed to be negative, whence both integral formulas would be infinite, let us use the reduction given in lemma I

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{1}{a}x^a(1-x^b)^{\frac{c}{b}} + \frac{c}{b}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1}$$

and after the application to the first integral formula by taking $c = -a-\alpha$ it will be

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = \frac{1}{a}x^a(1-x^b)^{\frac{-a-\alpha}{b}} - \frac{a+\alpha}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{-a-\alpha-b}{b}}.$$

But for our other integral formula one has to take $c = -a$ and it will be

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{1}{a}x^a(1-x^b)^{\frac{-a}{b}} - \int x^{a+b-1}\partial x(1-x^b)^{\frac{-a-b}{b}}.$$

Because the exponent $a+b$ is already positive, by means of the reduction in corollary II, which was

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a+c}{c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}},$$

for the case of the last formula we will have

$$\int x^{a+b-1}\partial x(1-x^b)^{\frac{-a-b}{b}} = -\frac{b}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{-a}{b}} = -\frac{b}{a}\Delta'$$

and so our last integral formula will be expressed in such a way that it is

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{1}{a}x^a(1-x^b)^{\frac{-a}{b}} + \frac{b}{a}\Delta',$$

which value subtracted from the first formula gives this expression for the left-hand side

$$\frac{1}{a}x^a(1-x^b)^{\frac{-a-\alpha}{b}} - \frac{a+\alpha}{a}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} - \frac{1}{a}x^a(1-x^b)^{\frac{-a}{b}} - \frac{b}{a}\Delta'.$$

Here certainly, since a is supposed to be negative, both absolute terms become infinite having put $x = 0$; but the two combined are represented this way

$$\frac{1}{a}x^a(1-x^b)^{\frac{-a}{b}}\left((1-x^b)^{\frac{-\alpha}{b}-1}\right),$$

which form for an infinitely small x because of

$$(1-x^b)^{\frac{-\alpha}{b}} = 1 + \frac{\alpha}{b}x^b + \text{etc.}$$

is transformed into this one

$$\frac{\alpha}{ab}x^{a+b}(1-x^b)^{\frac{-a}{b}},$$

which because of $a + b > 0$ having put $x = 0$ certainly vanishes, as the condition of the integration demands it. But having put $x = 1$ the whole absolute term also vanishes; therefore, for the right-hand side of our equation we will have

$$-\frac{b}{a}\Delta' - \frac{a+\alpha}{a}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}};$$

if the left-hand side $\frac{b}{a}\Delta'(S-1)$ is set equal to this, we will obtain this value

$$S = -\frac{a+\alpha}{b\Delta'}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}},$$

which expression already holds for all cases, in which $a + b$ is a positive number.

COROLLARY 1

§24 Since in the preceding paragraph we found

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}} = -\frac{b}{a}\Delta',$$

it is known that the value of this integral formula extended from $x=0$ to $x=1$ is reduced to this form

$$\frac{\pi}{b\sin\frac{(a+b)}{b}\pi},$$

whence the quantity contained in the character Δ' becomes known, which will be

$$\Delta' = -\frac{-\pi a}{bb\sin\frac{(a+b)}{b}\pi},$$

which value is reduced this way

$$\Delta' = \frac{\pi a}{bb\sin\frac{a\pi}{b}}.$$

COROLLARY 2

§25 But now let us substitute the integral formula for Δ' again and the found sum S will be expressed this way by means of two integral formulas

$$S = -\frac{a+\alpha}{b}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} : \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}},$$

which can also be expressed this way

$$S = \frac{a+\alpha}{a}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} : \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}},$$

which expression therefore holds, whenever it is $a+b>0$, even though a might be negative; but on the other hand for the cases, in which the exponent a itself is positive, for the same series we found the sum

$$S = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} : \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}}.$$

COROLLARY 3

§26 If we consider these two forms more accurately, we will quickly discover that the form found here can easily be derived form the preceding one by means of the reduction shown in the corollary of the first lemma, where it was

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \frac{c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1}.$$

For, if we apply this reduction to the form found above for S, for the numerator it will be $c = -a - \alpha$, whence the numerator is transformed this way

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = -\frac{a+\alpha}{a}\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}}.$$

Further, on the other hand it will be $c = -a$ for the denominator and hence the denominator itself

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}} = -\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}};$$

here it is evident, if the numerator is divided by the denominator, that the value we obtained in this problem results.

SCHOLIUM

§27 Therefore, although the expression found above for the summation of the series

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \text{etc.},$$

which is follows

$$S = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} : \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}},$$

is only valid for the cases, in which it is $a > 0$, nevertheless from it one would have easily been able to deduce the expression for S found here, which even holds, as long as $a + b > 0$, which we obtained here not without long detours; but on the other hand here the reason is clearly understood, why it is possible to use such a reduction, although the reduction given in § 7 can only be valid, if the exponent a was positive, since the absolute part is neglected, whence the reduction so of the numerator as of the denominator considered separately would be erroneous; but both errors, committed so in the numerator as in the denominator, luckily compensate for each other. Therefore, we will be able to use this new method safely, whenever even larger negative values are attributed to the exponent a.

PROBLEM 6

§28 *The capital letters, so the Latin as The Germanic ones, shall retain the same values we assigned to them the whole time; to define the sum of the series*

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \mathfrak{D}D + \text{etc.},$$

whenever the exponent a obtains arbitrarily large negative values.

SOLUTION

For the cases, in which the exponent a is positive, the sum of this series is expressed in such a way that it is

$$S = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} : \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}}.$$

Furthermore, for negative values of a, if it only was $a+b>0$, by means of the reduction in § 7 we just found

$$S = \frac{a+\alpha}{a} \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} : \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}}.$$

But if just the formula $a+2b$ was positive, let us apply the explained reduction to the closest following formula, and one has to take $a = a+b$ and $c = -a-\alpha-b$ for the numerator, but for the denominator on the other hand $c = -a-b$; hence one finds

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} = -\frac{a+\alpha+b}{a+b} \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+2b}}}$$

and

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}} = -\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+2b}}};$$

having substituted these values in the last expression for S for the case $a+2b>0$ we will find

$$S = \frac{a+\alpha}{a} \cdot \frac{a+\alpha+b}{a+b} \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+2b}}} : \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+2b}}}.$$

But if just $a+3b$ obtains a positive value, a similar reduction will lead to the following expression

$$S = \frac{a+\alpha}{a} \cdot \frac{a+\alpha+b}{a+b} \cdot \frac{a+\alpha+2b}{a+2b} \int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+3b}}} : \int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+3b}}}.$$

In like manner, if just the formula $a+4b$ has a positive value, the sum in question will be found to be

$$S = \frac{a+\alpha}{a} \cdot \frac{a+\alpha+b}{a+b} \cdot \frac{a+\alpha+2b}{a+2b} \cdot \frac{a+\alpha+3b}{a+3b} \int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+4b}}} : \int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+4b}}}.$$

In all these formulas the denominators admit it to be reduced to a circular arc. For, since the general form of the denominators is

$$\int \frac{x^{a+bn-1}\partial x}{\sqrt[b]{(1-x^b)^{a+nb}}},$$

by means of the things shown above it is plain that its value is

$$\frac{\pi}{b \sin\left(\frac{a+nb}{b}\pi\right)};$$

here, since it is

$$\sin\left(\frac{a+nb}{b}\right)\pi = \sin\left(n\pi + \frac{a}{b}\pi\right),$$

it is evident that in the cases, in which n is an even number, the denominator will be

$$= \frac{\pi}{b \sin\frac{a\pi}{b}};$$

but in cases, in which n is an odd number, the denominator will be

$$= \frac{-\pi}{b \sin\frac{a\pi}{b}}.$$

Furthermore, all these formulas are to be considered to be completely identical to each other, since all are deduced from the first by means of the reductions given above, if only the absolute parts are neglected. For, the power of these formulas is so large, that, even though these reductions so for the numerator as for the denominator taken separately would be false, these two errors nevertheless cancel each other.

Example

§29 Let us take $b = 2$ and let $a = -5$ and $\alpha = -4$, whence this series results

$$(1-xx)^{\frac{5}{2}} = 1 - \frac{5}{2}xx + \frac{5\cdot 3}{2\cdot 4}x^4 - \frac{5\cdot 3\cdot 1}{2\cdot 4\cdot 6}x^6 - \frac{5\cdot 3\cdot 1\cdot 1}{2\cdot 4\cdot 6\cdot 8}x^8 - \frac{5\cdot 3\cdot 1\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 8\cdot 10}x^{10} - \text{etc.}$$

and

$$(1-xx)^2 = 1 - 2xx + x^4,$$

where the series to be summed will be

$$S = 1 + 5 + \frac{15}{8} = \frac{63}{8}.$$

Since here just $1+3b$ becomes a positive quantity, namely $a + 3b = 1$, one will have to use the second formula, whence one concludes

$$S = \frac{9\cdot 7\cdot 5}{5\cdot 3\cdot 1}\int \frac{\partial x}{\sqrt{(1-xx)^{-3}}} : \int \frac{\partial x}{\sqrt{1-xx}},$$

where the denominator is

$$\int \frac{\partial x}{\sqrt{1-xx}} = \frac{\pi}{2};$$

but the numerator on the other hand is

$$\int \partial x \sqrt{(1-xx)^3},$$

which formula by means of the first reduction in § 10, having put $a = 1$ and $b = 2$, yields

$$\int \partial x (1-xx)^{\frac{c}{2}} = \frac{c}{1+c}\int \partial x (1-xx)^{\frac{c}{2}-1},$$

whence because of $c = 3$ it will be

$$\int \partial x (1-xx)^{\frac{3}{2}} = \frac{3}{4}\int \partial x (1-xx)^{\frac{1}{2}}.$$

But further, since here it is $c = 1$, it will be

$$\int \partial x(1-xx)^{\frac{1}{2}} = \frac{1}{2}\int \frac{\partial x}{\sqrt{1-xx}} = \frac{\pi}{4},$$

whence we will have for the numerator

$$\int \partial x(1-xx)^{\frac{3}{2}} = \frac{3\pi}{16};$$

having substituted these values the sum in question results as

$$S = \frac{9\cdot 7\cdot 5}{5\cdot 3\cdot 1}\cdot\frac{3\pi}{16} : \frac{\pi}{2} = \frac{63}{8},$$

which is true.

Another Method to find the sums of the same Series

§30 We found the preceding sums from the nature of the series itself; for, the single terms are products of two binomial coefficients. But since not so long ago I demonstrated[7], if the series is formed in such a way, that it is

$$S = 1 + \left(\frac{m}{1}\right)\left(\frac{n}{1}\right) + \left(\frac{m}{2}\right)\left(\frac{n}{2}\right) + \left(\frac{m}{3}\right)\left(\frac{n}{3}\right) + \left(\frac{m}{4}\right)\left(\frac{n}{4}\right) + \text{etc.},$$

that it will then be

$$S = \left(\frac{m+n}{m}\right) \quad \text{or even} \quad \left(\frac{m+n}{n}\right),$$

whose value expanded in usual manner yields

$$S = \frac{m+n}{1}\cdot\frac{m+n-1}{2}\cdot\frac{m+n-2}{3}\cdot\frac{m+n-3}{4}\cdots\frac{m+1}{n},$$

this value, as often as m and n are integer numbers, can hence always be assigned without any difficulty. But whenever one takes fractions for these numbers, I exhibited the value of this expression in the following way in terms of integral formulas, that it is

$$S = \frac{\int u^{m+n}\partial x}{\int u^m\partial x\cdot\int u^n\partial x},$$

while $u = \log\frac{1}{x}$, but then having extended the integral from $x = 0$ to $x = 1$.

[7]Euler refers to E575 again.

§31 But this reduction does not seem to be useful for our undertaking, since it involves quadratures of transcendental curves; but on the other hand considering it with more attention I found that the value of the same formula $\left(\frac{m+n}{m}\right)$ can also be reduced to quadratures of algebraic curves, which turned out to be even simpler than those we obtained by the preceding method and which do not suffer from the inconvenience that for different exponents other reductions have to be used every time. Therefore, I will explain this new method here more clearly.

§32 But this method is deduced from the reductions mentioned above in § 12, where we put

$$\Delta = \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1}.$$

But here let us immediately take $b=1$ and $a=1$, so that it is

$$\Delta = \int \partial x(1-x)^{c-1} = \frac{1}{c};$$

but then the reductions mentioned in § 12 will be as follows

$$\begin{aligned}
\int x \;\; \partial x(1-x)^{c-1} &= \frac{1}{c+1}\Delta,\\
\int xx\partial x(1-x)^{c-1} &= \frac{1}{c+1}\cdot\frac{2}{c+2}\Delta,\\
\int x^3\,\partial x(1-x)^{c-1} &= \frac{1}{c+1}\cdot\frac{2}{c+2}\cdot\frac{3}{c+3}\Delta,\\
\int x^4\,\partial x(1-x)^{c-1} &= \frac{1}{c+1}\cdot\frac{2}{c+2}\cdot\frac{3}{c+3}\cdot\frac{4}{c+4}\Delta
\end{aligned}$$

etc.,

whence it is concluded that in general it will be

$$\int x^\lambda\partial x(1-x)^{c-1} = \frac{1}{c+1}\cdot\frac{2}{c+2}\cdot\frac{3}{c+3}\cdots\frac{\lambda}{c+\lambda}\Delta.$$

§33 If now the last formula is inverted, one will find

$$\frac{\Delta}{\int x^\lambda\partial x(1-x)^{c-1}} = \frac{c+1}{1}\cdot\frac{c+2}{2}\cdot\frac{c+3}{3}\cdot\frac{c+4}{4}\cdots\frac{c+\lambda}{\lambda};$$

but this product, if the numerators are written in reverse order, will obtain this form
$$\frac{c+\lambda}{1}\cdot\frac{c+\lambda-1}{2}\cdot\frac{c+\lambda-2}{3}\cdot\frac{c+\lambda-3}{4}\cdots\frac{c+1}{\lambda},$$
whence, because the sum in question $S=\left(\frac{m+n}{n}\right)$ in expanded form gave

$$S=\frac{m+n}{1}\cdot\frac{m+n-1}{2}\cdot\frac{m+n-2}{3}\cdots\frac{m+1}{1},$$

that form is obviously transformed into this one, by taking $c=m$, $\lambda=n$, from which it is $\Delta=\frac{1}{m}$, and the sum in question itself will be expressed this way

$$S=\frac{1}{m\int x^n\partial x(1-x)^{m-1}},$$

and because it is possible to permute the two numbers m and n, it will also be

$$S=\frac{1}{n\int x^m\partial x(1-x)^{n-1}}.$$

§34 This expression, as often as m and n are integer numbers, obviously yields the true sum. For the sake of an example, let it be $m=4$ and $n=3$, and since

$$(1+z)^4+1+4z+6zz+4z^3+z^4$$

and

$$(1+z)^3=1+3z+3zz+z^3,$$

the series to be summed will be

$$S=1+3\cdot 4+3\cdot 6+1\cdot 4=35.$$

But on the other hand by means of the first integral formula we have

$$S=\frac{1}{4\int x^3\partial x(1-x)^3},$$

but by means of the last formula it will be

$$S=\frac{1}{3\int x^4\partial x(1-x)^2}.$$

But for the first it is
$$\int x^3 \partial x (1-x)^3 = \frac{1}{140},$$
for the other one
$$\int x^4 \partial x (1-x)^2 = \frac{1}{5} - \frac{2}{6} + \frac{1}{7} = \frac{1}{105},$$
so that from each of both formulas $S = 35$ results.

§35 In the reductions, from which we derived these expressions, we assumed the integrals to be taken in such a way that they are extended from the lower limit $x = 0$ to the upper limit $x = 1$. But here conveniently the same circumstance can be used which we observed in the preceding solution, namely, that the formula found here also holds, even though the exponents were negative, in which cases it is certainly not possible to observe the used reduction; for, here also two errors cancel each other. So, if it was $m = -4$ and $n = 3$, it will be
$$(1+z)^{-4} = 1 - 4z + 10zz - 20z^3 + 35z^4 - 56z^5 + \text{etc.}$$
and
$$(1+z)^3 = 1 + 3z + 3zz + z^3,$$
and so the series to be summed will be
$$S = 1 - 3 \cdot 4 + 3 \cdot 10 - 1 \cdot 20 = -1.$$
But on the other hand the last integral formula gives
$$S = \frac{1}{3 \int \frac{\partial x}{x^4}(1-x)^2}.$$
But it is
$$\int \frac{\partial x}{x^4}(1-x)^2 = \frac{-1}{3x^3} + \frac{2}{2xx} - \frac{1}{x},$$
which expression vanishes having put $x = \infty$, but having put $x = 1$ it gives $-\frac{1}{3}$. But if both numbers m and n are assumed to be negative, the sum of the series would obviously become infinite.

§36 But because these cases, in which m and n are integer numbers, do not cause any difficulty, we will still be able to apply our principal formula, whenever the numbers m and n are fractional numbers. So, if it was $m = \frac{1}{2}$ and $n = \frac{1}{2}$, because of

$$(1+z)^{\frac{1}{2}} \quad = 1 + \frac{1}{2}z - \frac{1 \cdot 1}{2 \cdot 4}zz + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}z^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}z^4 + \text{etc.}$$

and

$$(1+z)^{-\frac{1}{2}} = 1 - \frac{1}{2}z + \frac{1 \cdot 3}{2 \cdot 4}zz - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}z^3 + \frac{1 \cdot 3 \cdot 5 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}z^4 - \text{etc.}$$

the sum to be summed will be

$$S = 1 - \frac{1}{2} \cdot \frac{1}{2} - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{1 \cdot 3}{2 \cdot 4} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} - \text{etc.}$$

But on the other hand the first integral formula gives us

$$S = \frac{2}{\int \frac{\partial x}{\sqrt{x(1-x)}}}.$$

But now having put $x = yy$ it is

$$\int \frac{\partial x}{\sqrt{x(1-x)}} = 2\int \frac{\partial y}{\sqrt{1-yy}} = \pi,$$

so that hence it is $S = \frac{2}{\pi}$, which sum we found already above [§ 21] for the same case.

§37 For other cases it will be especially helpful to use the reductions explained above; in order to do so more easily, let us consider this general form

$$\int x^q \partial x (1-x)^r;$$

and those six reductions mentioned above in the corollaries of the lemmas will yield the following equations

$$\text{I.}\quad \int x^q \partial x(1-x)^r = \frac{r}{q+1}\int x^{q+1}\partial x(1-x)^{r-1},$$

$$\text{II.}\quad \int x^q \partial x(1-x)^r = \frac{q}{r+1}\int x^{q-1}\partial x(1-x)^{r+1},$$

$$\text{III.}\quad \int x^q \partial x(1-x)^r = \frac{q+r+2}{q+1}\int x^{q+1}\partial x(1-x)^{r},$$

$$\text{IV.}\quad \int x^q \partial x(1-x)^r = \frac{q}{q+r+1}\int x^{q-1}\partial x(1-x)^{r},$$

$$\text{V.}\quad \int x^q \partial x(1-x)^r = \frac{r}{q+r+1}\int x^{q}\quad\partial x(1-x)^{r-1},$$

$$\text{VI.}\quad \int x^q \partial x(1-x)^r = \frac{q+r+2}{r+1}\int x^{q}\quad\partial x(1-x)^{r+1}.$$

§38 By means of these reductions the first expression [§ 33] found for the sum

$$S = \frac{1}{m\int x^n\partial x(1-x)^{m-1}},$$

where it is $q = n$ and $r = m-1$, can be cast into the following six formulas:

$$\text{I.}\quad S = \frac{n+1}{m(m-1)\int x^{n+1}\partial x(1-x)^{m-2}},$$
$$\text{II.}\quad S = \frac{1}{n\int x^{n-1}\partial x(1-x)^{m}},$$
$$\text{III.}\quad S = \frac{n+1}{m(m+n+1)\int x^{n+1}\partial x(1-x)^{m-1}},$$
$$\text{IV.}\quad S = \frac{m+n}{mn\int x^{n-1}\partial x(1-x)^{m-1}},$$
$$\text{V.}\quad S = \frac{m+n}{m(m-1)\int x^{n}\partial x(1-x)^{m-2}},$$
$$\text{VI.}\quad S = \frac{1}{(m+n+1)\int x^{n}\partial x(1-x)^{m}},$$

which same forms also follow from the last.

§39 But these formulas can always be used in such a way that in the integral formulas both exponents of x and of $1-x$ lie within the boundaries 0 and -1, which formulas are usually especially considered. So, if it was $m = \frac{7}{2}$ and $n = 4$, hence it will be

$$(1+z)^{\frac{7}{2}} = 1 + Az + Bzz + Cz^3 + \text{etc.}$$

and

$$(1+z)^4 = 1 + 4z + 6zz + 4z^3 + z^4,$$

so that the series to be summed is

$$S = 1 + 4A + 6B + 4C + D;$$

but on the other hand it will be

$$S = \frac{2}{7\int x^4\partial x(1-x)^{\frac{5}{2}}},$$

where it is $q = 4$ and $r = \frac{5}{2}$. Therefore, first we will be able to push the exponent q down to zero, which happens by means of reduction IV; for, hence it will be

$$\int x^4\partial x(1-x)^{\frac{5}{2}} = \frac{8}{15}\int x^3\partial x(1-x)^{\frac{5}{2}},$$

but further

$$\int x^3\partial x(1-x)^{\frac{5}{2}} = \frac{6}{13}\int xx\partial x(1-x)^{\frac{5}{2}},$$

then

$$\int xx\partial x(1-x)^{\frac{5}{2}} = \frac{4}{11}\int x\partial x(1-x)^{\frac{5}{2}},$$

finally

$$\int x\partial x(1-x)^{\frac{5}{2}} = \frac{2}{9}\int \partial x(1-x)^{\frac{5}{2}};$$

and so we will now have

$$S = \frac{2\cdot 15\cdot 13\cdot 11\cdot 9}{7\cdot 8\cdot 6\cdot 4\cdot 2\int \partial x(1-x)^{\frac{5}{2}}}.$$

§40 Since further r is $=\frac{5}{2}$, this exponent will be lowered by means of reduction V, whence because of $q=0$ and $r=\frac{5}{2}$ it is

$$\int \partial x(1-x)^{\frac{5}{2}} = \frac{5}{7}\int \partial x(1-x)^{\frac{3}{2}};$$

in like manner it will be

$$\int \partial x(1-x)^{\frac{3}{2}} = \frac{3}{5}\int \partial x(1-x)^{\frac{1}{2}},$$

and finally

$$\int \partial x(1-x)^{\frac{1}{2}} = \frac{1}{3}\int \frac{\partial x}{\sqrt{1-x}},$$

from which it is concluded

$$S = \frac{2\cdot 15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3}{7\cdot 8\cdot 6\cdot 4\cdot 2\cdot 5\cdot 3\cdot 1\int \frac{\partial x}{\sqrt{1-x}}}.$$

On the other hand it is

$$\int \frac{\partial x}{\sqrt{1-x}} = 2-2\sqrt{1-x},$$

and so its value will be $=2$, and having done the calculation one will find $S=\frac{6435}{128}$.

§41 Since now it was $m = \frac{7}{2}$, it will be

$$A = \frac{7}{2}, \quad B = \frac{7 \cdot 5}{2 \cdot 4}, \quad C = \frac{7 \cdot 5 \cdot 3}{2 \cdot 4 \cdot 6}, \quad D = \frac{7 \cdot 5 \cdot 3 \cdot 1}{2 \cdot 4 \cdot 6 \cdot 8},$$

so that our series to be summed is

$$S = 1 + 14 + \frac{105}{4} + \frac{35}{4} + \frac{35}{128} = \frac{6435}{128},$$

which extraordinarily agrees with the sum found before.

§42 Further, concerning the series we denoted by the general term $\left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right)$, where in order the numbers 0, 1, 2, 3, 4 etc. are to be written instead of x, so that it is

$$S = \int \left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right),$$

I showed that it will be

$$S = \left(\frac{m+n}{m+p}\right) \quad \text{or even} \quad S = \left(\frac{m+n}{n-p}\right);$$

hence it is plain that this sum will be the same as if the propounded series would have been $\int \left(\frac{m+p}{x}\right)\left(\frac{n-p}{x}\right)$. Therefore, our formulas given above will be accommodated to this sum, if in them these values $m+p$ and $n-p$ are written instead of the letters m and n, and so from the first formula this sum will be

$$S = \frac{1}{(m+p)\int x^{n-p}\partial x(1-x)^{m+p-1}},$$

but from the second it will be

$$S = \frac{1}{(n-p)\int x^{m+p}\partial x(1-x)^{n-p-1}},$$

and so this whole argument is to be considered to be completed.

§43 It will be worth one's while to have added the case here, in which it is $m+n=1$ and hence $m=1-n$; and the sum of the series from formula IV, § 38 will be

$$S=\frac{1}{n(1-n)\int\frac{x^{n-1}\partial x}{(1-x)^n}},$$

which integral can conveniently be expressed by means of circular arcs; for, it will be

$$\int\frac{x^{n-1}\partial x}{(1-x)^n}=\frac{\pi}{\sin n\pi},$$

so that the sum of the propounded series is

$$S=\frac{\sin n\pi}{mn\pi}.$$

Hence, if it was $m=\frac{1}{2}$ and $n=\frac{1}{2}$, it will be $S=\frac{4}{\pi}$, which is the sum of the series

$$1+\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot 1}{2\cdot 4}\right)^2+\left(\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\right)^2+\left(\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\right)^2+\text{etc.},$$

as we noted above already. Further, if we take $m=\frac{1}{3}$ and $n=\frac{2}{3}$, because of

$$(1+z)^{\frac{1}{3}}=1+\frac{1}{3}z-\frac{1\cdot 2}{3\cdot 6}zz+\frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}z^3-\frac{1\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}+\text{etc.}$$

and

$$(1+z)^{\frac{2}{3}}=1+\frac{2}{3}z-\frac{2\cdot 1}{3\cdot 6}zz+\frac{2\cdot 1\cdot 4}{3\cdot 6\cdot 9}z^3-\frac{2\cdot 1\cdot 4\cdot 7}{3\cdot 6\cdot 9\cdot 12}+\text{etc.}$$

the series to be summed will be

$$S=1+\frac{1}{3}\cdot\frac{2}{3}+\frac{1\cdot 2}{3\cdot 6}\cdot\frac{2\cdot 1}{3\cdot 6}+\frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}\cdot\frac{2\cdot 1\cdot 4}{3\cdot 6\cdot 9}+\frac{1\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}\cdot\frac{2\cdot 1\cdot 4\cdot 7}{3\cdot 6\cdot 9\cdot 12}+\text{etc.},$$

whose sum because of

$$\sin\frac{1}{3}\pi=\frac{\sqrt{3}}{2}$$

can be expressed by means of the quadrature of the circle, and it will be

$$S = \frac{9\sqrt{3}}{4\pi}.$$

If in like manner we take

$$m = \frac{1}{4} \quad \text{and} \quad n = \frac{3}{4},$$

the series to be summed will be

$$S = 1 + \frac{1}{4} \cdot \frac{3}{4} + \frac{1 \cdot 3}{4 \cdot 8} \cdot \frac{3 \cdot 1}{4 \cdot 8} + \frac{1 \cdot 3 \cdot 7}{4 \cdot 8 \cdot 12} \cdot \frac{3 \cdot 1 \cdot 5}{4 \cdot 8 \cdot 12} + \frac{1 \cdot 3 \cdot 7 \cdot 11}{4 \cdot 8 \cdot 12 \cdot 16} \cdot \frac{3 \cdot 1 \cdot 5 \cdot 9}{4 \cdot 8 \cdot 12 \cdot 16} + \text{etc.},$$

whose sum, because of

$$\sin \frac{1}{4}\pi = \frac{1}{\sqrt{2}},$$

will be

$$S = \frac{8\sqrt{2}}{3\pi};$$

but this series can be exhibited succinctly this way

$$S = 1 + \frac{3}{4^2}\left(1 + \frac{1 \cdot 3}{8^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{8^2 \cdot 12^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{8^2 \cdot 12^2 \cdot 16^2} + \text{etc.}\right).$$